PHYSICAL REVIEW E VOLUME 62, NUMBER 5 NOVEMBER 2000

Scaling behavior in a nonlocal and nonlinear diffusion equation

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(Received 22 July 1999; revised manuscript received 10 May 2000)

We present the results of analytical studies of a one-dimensional nonlocal and nonlinear diffusion equation describing nonequilibrium processes ranging from aggregation phenomena to the cooperation of individuals. On tuning the initial conditions, a dynamical transition with a universal scaling behavior is observed between two different asymptotic (in time) solutions. The scaling behavior at the transition is also obtained in a self-organized manner, independent of the initial conditions, on temporally evolving the diffusion equation subjected to a mirror symmetry transformation.

PACS number(s): 05.45.−a, 05.40.−a

In this Rapid Communication, we present a study concerning a dynamical transition in a nonlocal and nonlinear diffusion equation. The equation is essentially one describing biased diffusion [1] with the magnitude of the bias determined by the instantaneous spatial distribution of the walkers undergoing diffusion. The nonlocal nature of our equation is of interest as a prototype of more complex and intrinsically nonlocal processes, such as the growth of thin films in the presence of shadowing [2] and the sculpting of the drainage basin of river networks due to erosional processes [3]. The model exhibits a dynamical transition between two asymptotic states corresponding to the walkers moving ballistically in one direction or the other. We study the scaling behavior at the transition for a class of initial conditions, and find that the mean position of the walkers exhibits novel behavior and scales as the square root of the time. Strikingly, the peculiar scaling behavior of the transition is reached in a self-organized manner on modifying the equation by means of a mirror transformation.

Our equation generalizes a simple model [4] for emergence and evolution of cooperative behavior in groups of individuals, such as living beings or agents. This is an important topic in game theory, with fundamental applications to social and biological sciences [5]. In a recent paper, Sigmund and Nowak [4] showed that cooperation can arise even when a potential recipient has no chance of encountering his helper and directly return the help; this new mechanism is called ''indirect reciprocity.'' It relies on the altruistic behavior which increases the reputation of the donor in the community and, in turn, individuals with high reputation are more likely to be helped in the future. The reputation can be mathematically defined as an image-score x associated with each player. The image-score x is known to other members of the community and indicates both whether an individual provides help and if she is worthy of being given help. Sigmund and Nowak [4] considered a situation in which x is an integer number in $[-\infty,\infty]$ and a potential donor and a recipient of an altruistic act have an interaction in which the donor helps the recipient provided the recipient's image score x is positive. Such an altruistic act increases the image score of the donor by 1 (the selfish act would have decreased it by 1) and the image score of the recipient is unchanged. The time evolution of the fraction P_x of players with image score x is governed by the equation

$$P_x(t+1)=\frac{P_x(t)}{2}+\frac{\Phi(t)}{2}P_{x-1}(t)+\frac{1-\Phi(t)}{2}P_{x+1}(t), \tag{1}$$

with $\Phi(t)=\Sigma_{x\geqslant 0}P_x(t)$ representing the fraction of players with a non-negative image score at time t.

If all players start with a non-negative (negative) image score then they will always cooperate (defect) in the future. Therefore, from such initial conditions, only two asymptotic states emerge corresponding to ''all cooperators'' or ''all defectors.'' An interesting and nontrivial situation arises when the initial distribution of image scores has nonzero weight on both sides of the origin. In fact, there exists a threshold, Φ_c, for the initial value of $\Phi(t)$ such that only if $\Phi_0=\Phi(0)$ exceeds Φ_c, the system evolves towards total cooperation, otherwise it flows towards a state filled with defectors. The value of Φ_c clearly depends on the class of distributions chosen as the initial condition.

In a scheme, in which the players become random walkers in the space of image scores, we propose the basic continuum equation for $P(x,t)$, the probability that a walker occupies the position x at time t,

$$\frac{\partial P}{\partial t}=-v(t)\frac{\partial P}{\partial x}+\frac{1}{4}\frac{\partial^2 P}{\partial x^2}, \tag{2}$$

where both the nonlinearity as well as the nonlocality are introduced in the bias velocity v defined by

$$v(t)=\int_0^\infty dx\, P(x,t)-\frac{1}{2}=\Phi(t)-\frac{1}{2}. \tag{3}$$

Equation (2) is derived from Eq. (1) to leading order in the continuum limit of both time and image-score, the latter interpreted as a spatial coordinate.

On setting v equal to zero (i.e., $\Phi=1/2$) in Eq. (2), one recovers the standard unbiased diffusion equation [1], whereas one obtains simple biased diffusion, when v is a constant. In our equation, v is a direct measure of the imbalance between the population of walkers in the right and left, and the drift bias promotes further aggregation. Equation (2) describes the temporal evolution of the distribution function $P(x,t)$ and leads to one of two outcomes in the large time limit. Depending on the initial distribution, one ends up at long times with the bias to the right or to the left winning, so that P becomes 1 either at $x=+\infty$ or at $x=-\infty$. Our focus is on studying the transition (on varying the initial conditions) between these two limiting states, and we investigate the scaling behavior of the transient regime occurring before the system collapses onto the asymptotic states at $x=\pm\infty$.

Defining the new variables

$$w(t)=\int_0^t d\tau\, v(\tau), \quad y(t)=x-w(t), \tag{4}$$

Eq. (2) may be cast in the form of a standard diffusion equation,

$$\frac{\partial P}{\partial t}=\frac{1}{4}\frac{\partial^2 P}{\partial y^2}. \tag{5}$$

If one considers an initial Gaussian distribution centered around x_0 and with variance σ_0, the formal solution of Eq. (5) is given by

$$P(x,t)=N\exp\left\{-\frac{[x-x_0-w(t)]^2}{t+2\sigma_0^2}\right\}, \tag{6}$$

where $N=1/\sqrt{\pi(t+2\sigma_0^2)}$ is the normalization factor.

With $\sigma_0=0$, $P(x,t)$ is also the fundamental solution, which will be used to obtain the distribution at time t, starting from more general initial conditions $P_0(x)$, i.e.,

$$P(x,t)=\int_{-\infty}^{+\infty}\frac{dy}{\sqrt{\pi t}}P_0(y)\exp\left\{-\frac{[x-y-w(t)]^2}{t}\right\}. \tag{7}$$

However, expression (7) is only a formal solution of Eq. (2) because $w(t)$ is itself a function of $P(x,t)$ due to Eq. (3). Combining Eqs. (3) and (4) one obtains a self-consistent equation for the velocity $v=\dot{w}$,

$$\dot{w}=\tfrac{1}{2}\int_{-\infty}^{+\infty}dx\; P_0(x)\mathrm{erf}\left\{\frac{x+w(t)}{\sqrt{t}}\right\}. \tag{8}$$

The transition on varying the initial conditions is between two asymptotic (in time) states corresponding to aggregation on the right or on the left and therefore one expects that the transition point would correspond to a situation with no asymptotic bias [i.e., $v(\infty)=0$]. In this case, the asymptotic distribution is perfectly balanced with respect to the origin; there is nothing to choose between left and right and an unbiased behavior ensues. In order to explore the effect of the asymmetry in the initial distribution, let $P_0(x)=P_0^+(x)+P_0^-(x)$ where "+" and "−" indicate the even and odd part of P_0, respectively. For $P_0^-(x)=0$, symmetry considerations require that Eq. (8) admits only the solution $w(t)=0$, which implies that the velocity vanishes too. One can establish as well that, if $w(t)=0$, then necessarily $P_0(x)$ must be odd.

On assuming that the P_0^- component is tiny with respect to the even part, in a transient regime $|w(t)|$ remains small for all t less than a crossover time T_r, thus a perturbative approach may be implemented. To first order in P_0^-, we have

$$\dot{w}_1=\tfrac{1}{2}\int_{-\infty}^{+\infty}dx\; P_0^-(x)\mathrm{erf}(x/\sqrt{t})$$

$$+\frac{w_1}{\sqrt{\pi t}}\int_{-\infty}^{+\infty}dx\; P_0^+(x)\exp(-x^2/t). \tag{9}$$

The smaller the perturbation P_0^-, the longer is the time in which the perturbative regime holds. For initial distributions that are not too delocalized, $\mathrm{erf}(x/\sqrt{t})\sim 2x/\sqrt{\pi t}$ and $\exp(-x^2/t)\sim 1$, leading to the linearized equation

$$\dot{w}_1=\frac{1}{\sqrt{\pi t}}(a_-+a_+w_1), \tag{10}$$

where

$$a_-=\int_{-\infty}^{+\infty}dx\; P_0^-(x)x; \quad a_+=\int_{-\infty}^{+\infty}dx\; P_0^+(x).$$

a_- defines a measure of the asymmetry in P_0, while $a_+=1$ due to the normalization constraint.

This equation can be easily solved and yields

$$w_1(t)=a_-(e^{2\sqrt{t/\pi}}-1). \tag{11}$$

One may define a typical lifetime of the transient regime as the time, T_r, during which the linearization approximation holds. From Eq. (10), one finds that T_r scales as

$$T_r\sim\ln^2|a_-|. \tag{12}$$

The time scale to decide on one of the two different behaviors only diverges logarithmically as the asymmetry parameter a_- of the initial conditions vanishes. The system does not spend much time agonizing over which asymptotic state to select.

For simple initial distributions the asymmetry parameter a_- turns out to be proportional to or coincides with the quantity $\Phi_0-\Phi_c$.

For instance, when considering the Gaussian initial condition of Eq. (6), we see that $a_-\equiv x_0$ and it depends on Φ_0 through the relation

$$\Phi_0=\frac{1}{2}+\frac{1}{2}\,\mathrm{erf}\left(\frac{x_0}{\sigma_0\sqrt{2}}\right).$$

The condition $x_0=0$ corresponds to $\Phi_0=\Phi_c=1/2$. When $\Phi_0<\Phi_c$ ($\Phi_0>\Phi_c$) the population of walkers evolves, at large time, toward the state with $\Phi(t)=0$ [$\Phi(t)=1$]. Near this point one can expand the error function, obtaining $a_-=x_0\sim(\Phi_0-\Phi_c)$.

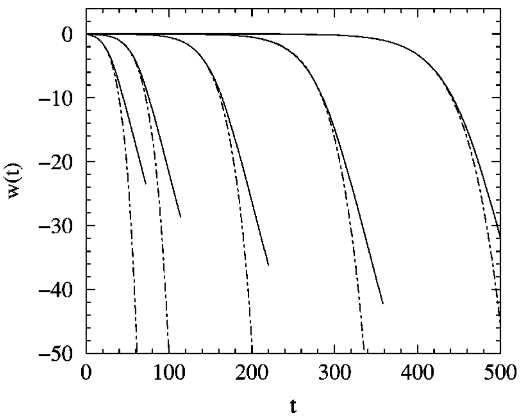

FIG. 1. Gaussian case: temporal behavior of $w_1(t)$ (dotted-dashed line) and the numerical solution of Eq. (8) (solid line) plotted for $\sigma=2$ and $x_0=-0.1,-10^{-2},-10^{-4},-10^{-8}$. T_r is defined as the time after which the solid and the corresponding dashed line differ in a noticeable manner.

In this Gaussian case, one can easily compare the solution $w(t)$ of Eq. (8) (numerically integrated) with the perturbative result $w_1(t)$, for different values of x_0. Figure 1 shows that $w(t)$ and $w_1(t)$ coincide within numerical errors up to a characteristic time T_r which grows with $|x_0|$ as predicted by Eq. (12)

$$T_r \sim \ln^2|\Phi_0-\Phi_c|. \tag{13}$$

The numerical data justifies, *a posteriori*, the use of perturbative theory to extract the scaling behavior of the transient lifetime T_r.

From dimensional analysis on Eq. (2) one expects that, in the transient regime, the typical velocity behaves like $v(t) \sim 1/\sqrt{T_r}$ in agreement with Eq. (10), while the characteristic length-scale ξ is expected to follow the diffusion law and therefore

$$\xi \sim \sqrt{T_r} \sim \big|\ln|\Phi_0-\Phi_c|\big|. \tag{14}$$

We have shown that the above logarithmic divergence of T_r and ξ in the transient regime holds for other families of initial conditions. For example, considering

$$P_0(x)=(1-\Phi_0)\,\delta(x-1)+\Phi_0\,\delta(x+1), \tag{15}$$

$\Phi_c=1/2$ is again the bifurcation point (unbiased, zero drift situation). The verification of the behavior (12) is shown in Fig. 2 (solid circles).

Equation (8) can be extended to the case of arbitrary initial conditions $P_0(x)$ with a fraction, f_0, of the walkers at $+\infty$. Recalling the general solution (7) we have

$$\dot{w}=\frac{f_0}{2}+\frac{1-f_0}{2}\int_{-\infty}^{\infty}dx\ P_0(x)\mathrm{erf}\left\{\frac{x+w(t)}{\sqrt{t}}\right\} \tag{16}$$

and

$$\Phi_0=f_0+(1-f_0)\int_0^{\infty}dx\ P_0(x). \tag{17}$$

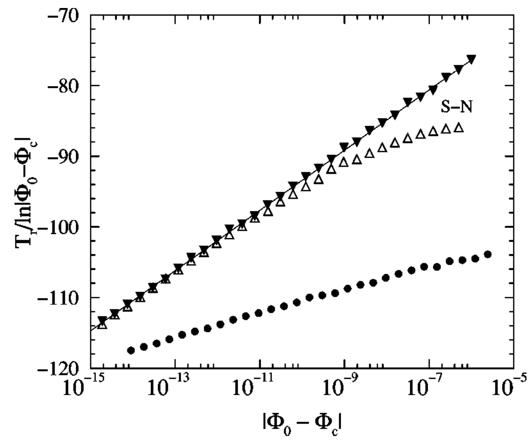

FIG. 2. Divergence of transient time T_r as a function of the deviation from the transition point $|\Phi_0-\Phi_c|$. Triangles refer to the *S-N* initial distribution (see text), when $\Phi_0<\Phi_c$ (open) and $\Phi_0>\Phi_c$ (closed). The bullets correspond to distribution (15) for $\Phi_c>\Phi_0$ (the data have been shifted to fit into the same scale). The predicted square logarithmic divergence corresponds to linear behavior in the plot.

Let us consider now an interesting lattice case [Eq. (1)], with initial condition: $P_0(+\infty)=\Phi_0$ and $P_0(x)=(1-\Phi_0)\,\delta_{x,-z}$. This is equivalent to introducing an absorbing boundary at $x=+\infty$ and $\Phi_0=f_0$. The bifurcation value of Φ_0 increases monotonically to a nonzero value $\Phi_c(z)$ for positive integer z with $\Phi_c(z\rightarrow\infty)=1/2$. The smallest value of $\Phi_c(z)$ for the discrete Eq. (2) occurs when $z=1$ (we denote this as the *S-N* distribution) which is the case we focus on in our numerical simulations. $\Phi_c(z=1)$ is found to be $0.261\,970\,531\,164\ldots$, a result that was noted earlier by Sigmund and Nowak [4]. Figure 2 shows a verification of Eq. (12) for $z=1$ (open and closed triangles).

At the bifurcation point, the average location of the random walkers (excluding the number fixed at $x=+\infty$) behaves asymptotically with time as

$$\langle x(t)\rangle \sim A\sqrt{t}, \tag{18}$$

with A a constant depending on f_0. This result can be predicted by an argument based on the asymptotic behavior of the continuum differential equation for w. Let us consider for instance Eq. (16), which admits two linear asymptotic solutions, $w_+(t)\sim t/2$ above Φ_c, and $w_-(t)\sim(f_0-1/2)t$, below Φ_c when $f_0<1/2$. Both behaviors correspond to regimes away from the bifurcation point. At the transition point Φ_c, however, the nonlinear drift exactly compensates the diffusion in such a way that $\dot{w}\rightarrow 0$ asymptotically. This constraint applied to Eq. (16) yields for the amplitude of Eq. (18)

$$\mathrm{erf}(A)=-f_0/(1-f_0). \tag{19}$$

The result (18) then follows on noting that the the average position of the walkers is linearly related to $w(t)$. We stress that this scaling regime is different from Eq. (10) because it holds strictly only at $\Phi_0=\Phi_c$.

The square-root behavior of $\langle x(t)\rangle$ crosses over to a linear regime when the bias reaches a sufficient strength. Numerically, there is indeed a sharp onset of the linear behavior at a value of $\langle x\rangle$ (see Fig. 3), which one may identify with ξ.

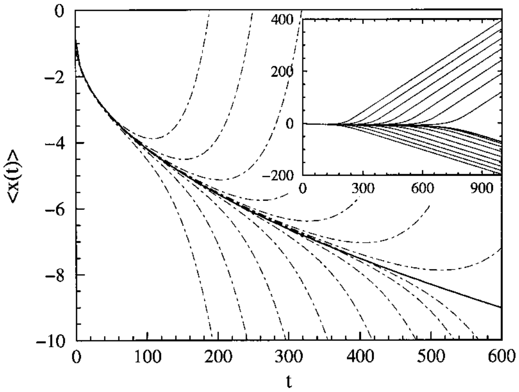

FIG. 3. Temporal behavior of the average location of walkers (excluding the fraction Φ_0 placed at $x=\infty$) for the *S-N* initial configuration. The upper (lower) dashed curves refer to a set of initial conditions with $\Phi_0 > \Phi_c$ ($\Phi_0 < \Phi_c$). The solid line indicates the best-fit for the envelope of the curves obtained through expression (18), with $A=-0.337$ and an additive constant $x_0=-0.847$. The figure vividly depicts the crossover of $\langle x(t)\rangle$ to a linear behavior, which may be used to define the characteristic length-scale ξ. The inset shows $\langle x(t)\rangle$ at larger times and the onset of the linear regime.

Indeed one can show that Eq. (16) admits as possible asymptotic solutions $w(t)\sim t$ and $w(t)\sim\sqrt{t}$. The former result pertains to the asymptotic behavior outside the transition regime, whereas the latter may be observed only at $\Phi=\Phi_c$. The sublinear behavior of $\langle x(t)\rangle$ is strikingly different from from the conventional diffusion, in which the rms displacement grows as the square root of time [6,7]. Here, on the other hand, the mean position of the walkers scales as the square root of time.

We now turn to study the effects of mirror symmetry $x\to -x$ on the scaling behavior of our model. In Eq. (2) the transformation $x\to -x$ is equivalent to a change in the sign of the bias velocity. [The discrete Eq. (1) is not completely invariant under mirror symmetry. In order to make it fully symmetric, one has to redefine the quantity $\Phi(t)=P_0/2+\Sigma_{x>0}P_x(t)$. In the continuum limit, this difference does not affect the scaling or the eventual asymptotic behavior.] In this situation, the system spontaneously organizes in such a way that the aggregation of walkers is disfavored. As a consequence, the asymptotic distribution approaches Eq. (6) with a fraction of the walkers at $+\infty$ and $\langle x(t)\rangle$ is given by Eq. (18) without any tuning of the initial conditions. This is the situation one would encounter with Eq. (2) when $\Phi_0=\Phi_c$ and $\Phi(t)$ approaches 1/2 asymptotically (i.e., perfectly balanced distribution). In this asymptotic regime, the scaling (18) of $\langle x(t)\rangle$ still ought to hold as confirmed by simulations performed on Eq. (1).

We have carried out further detailed simulations that complement our study. The temporal behavior of the bias velocity and the scaling of the characteristic length ξ, along with the self-organization to the transition state, are all found to be in excellent accord with our predictions.

This work was supported by INFN, NASA, and The Donors of the Petroleum Research Fund administered by the American Chemical Society.

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